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Summary

A shaping filter is a linear system which transforms stationary white noise into a possibly nonstationary random process having a given covariance $r(t,\tau)$. We show that stability of a shaping filter, appropriately defined, may be determined by inspection of $r(t,\tau)$. Stability is defined in the sense that any square-integrable input produces a bounded output (abbreviated L_2IBO).

The following results are proved:

1. A system with impulse response $h(t,\tau)$ is L_2IBO stable if and only if $\int_{t_o}^{t} h^2(t,\tau)d\tau \le C < \infty$ for all $t \ge t_o$; t_o is a fixed initial time (perhaps $-\infty$).
2. If $r(t,t) \le C < \infty$ for all $t \ge t_o$, the shaping filter is L_2IBO stable.
3. (Converse to 2) If the shaping filter is (uniformly) completely controllable, L_2IBO stability implies that $r(t,t) \le C < \infty$ for $-\infty$ $(\le)$ $< t_o \le t$.

Also included in the hypotheses of 3 is an assumption to the effect that the system is internally linear. The results of the paper apply to distributed as well as lumped systems. Detailed knowledge of the impulse response (or state-variable equations) is not required.

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Stability Properties of Shaping Filters

It is often of interest for various signal processing applications to determine a stable shaping filter for a covariance function $r(t, \tau)$, i.e., to determine a stable linear system that transforms stationary white noise into a random process having the convariance $r(t, \tau)$ [1], [2]. We show here that stability of a shaping filter, appropriately defined, may be determined by inspection of the covariance, provided that a shaping filter for $r(t, \tau)$ is known to exist.

Let $y(t)$ for $-\infty \leq t_0 \leq t < \infty$ be a zero-mean, possibly nonstationary random process with covariance

$$E[y(t)y(\tau)] = r(t, \tau) \tag{1}$$

defined everywhere; i.e., $r(t, \tau)$ is finite for finite t and τ. Suppose a nonanticipative shaping filter for $r(t, \tau)$ exists, having $h(t, \tau)$ as its impulse response. Then

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$$y(t) = \begin{cases} f(t, t_0, x_0) + \int_{t_0}^{t} h(t, \lambda)\omega(\lambda)\, d\lambda, & \text{for } t \geq t_0, \\ 0, & \text{for } t < t_0. \end{cases} \tag{2}$$

The function $\omega(\lambda)$ represents a stationary white noise input. The parameter x_0 assumes the values of intial conditions within the shaping filter, generally random variables uncorrelated with $\omega(\lambda)$, and may be regarded as the value of a state variable at $t = t_0$. The "transient response" $f(t, t_0, x_0)$ represents the output effect of the internal initial conditions.

An elementary but important consequence of (1) and (2) is that

$$r(t, \tau) = E[f(t, t_0, x_0)f(\tau, t_0, x_0)] + \int_{t_0}^{\min(t,\tau)} h(t, \lambda)h(\tau, \lambda)\, d\lambda, \tag{3}$$

which relation is used below to determine a stability property of the shaping filter.

Stability of a system is defined here in the sense that, beginning from the zero state, any square-integrable input produces a bounded output. Stability in this sense will be abbreviated as L₂IBO stability. This type of stability has not frequently been encountered in the linear system literature. However, if certain regularity conditions are imposed on the system coefficients or on the impulse responses, then L₂IBO stability is equivalent to the more familiar concept of bounded-input bounded-output (BIBO) stability [3], [4]. L₂IBO stability is formally defined below.

Definition 1

Let $z(t, u)$ represent the zero-state response of a linear system to the input $u(t)$ for $t_0 \leq t < \infty$. The system is L₂IBO stable if and only if for any $u(t)$ staisfying

$$\int_{t_0}^{\infty} u^2(t)\, dt \leq M_0 < \infty$$

for some finite M_0, there exists another constant M_1 (dependent upon M_0) such that $|z(t, u)| \leq M_1 < \infty$ for all $t \geq t_0$.

A necessary and sufficient condition for L₂IBO stability is established in the following theorem.

Theorem 1

A linear system with impulse response $h(t, \tau)$ is L₂IBO stable if and only if

$$\int_{t_0}^{t} h^2(t, \tau)\, d\tau \leq C < \infty, \qquad \text{for all} \quad t \geq t_0.$$

Proof of Sufficiency: Let $u(t)$ satisfy

$$\int_{t_0}^{\infty} u^2(t)\, dt \leq M < \infty.$$

Then from the Schwarz inequality,

$$\begin{aligned} z^2(t, u) &= \left[\int_{t_0}^{t} h(t, \tau)u(\tau)\, d\tau\right]^2 \\ &\leq \int_{t_0}^{t} h^2(t, \tau)\, d\tau \int_{t_0}^{t} u^2(\tau)\, d\tau \\ &\leq C \int_{t_0}^{\infty} u^2(\tau)\, d\tau \leq CM < \infty. \end{aligned}$$

Proof of Necessity: Let

$$\int_{t_0}^{t} h^2(t, \tau)\, d\tau$$

be unbounded. We will construct an input that produces an output exceeding any given bound. Let

$$C(t) = +\left[\int_{t_0}^{t} h^2(t, \tau)\, d\tau\right]^{1/2},$$

an unbounded function by assumption. Hence, for any $M > 0$ there is a $T > t_0$ such that $C(T) > M$. Holding T as a fixed parameter, define a sequence of inputs as

and note that

$$\int_{t_0}^{\infty} u_T^2(t)\, dt = C^2(T)/C^2(T) = 1,$$

for all $T > t_0$. The output $z(t, u_T)$ evaluated at $t = T$ is given by

$$z(T, u_T) = \int_{t_0}^{T} h^2(T, \tau)/C(T)\, d\tau = C(T) > M > 0.$$

Since M may be chosen arbitrarily large, the system is not L₂IBO stable.

Stability of a shaping filter is guaranteed by uniform boundedness of the covariance function, as is now demonstrated.

Theorem 2

Let a shaping filter for the covariance $r(t, \tau)$ exist[1] for all $t \geq t_0$. If $r(t, t) \leq C < \infty$ for all $t \geq t_0$, the shaping filter is L₂IBO stable.

Proof: From (3)

$$r(t, t) = E[f^2(t, t_0, x_0)] + \int_{t_0}^{t} h^2(t, \lambda)\, d\lambda. \tag{4}$$

Thus

$$\int_{t_0}^{t} h^2(t, \lambda)\, d\lambda \leq r(t, t) \leq C < \infty,$$

which, from Theorem 1, proves the assertion.

Note that the results above remain valid in the limit as t_0 approaches $-\infty$. Theorem 2 has a weak converse, which is proved below. This converse will be considered for two cases: finite t_0, and the limiting case as t_0 approaches $-\infty$. For this purpose, it is convenient to introduce the following three assumptions, which restrict the structure of systems in the class under consideration.

$S1$—For any x_0, t_0, and t_1, there is an x_1 such that $f(t, t_0, x_0) = f(t, t_1, x_1)$.

$C1$—The system is completely controllable at t_0 with respect to the parameter x. That is, given any x_1, there is a finite $t_1 > t_0$ with t_1 independent of x_1, such that the system may be transferred from the zero state at t_0 to the x_1 state at t_1 by application of some finite energy input $u(t)$ for $t_0 \leq t \leq t_1$.

$C2$—The system is uniformly completely controllable with respect to x. That is, for any t and x_1, there is a $t_1 > t$ for which $t_1 - t \leq T$, with T independent of t and x_1, such that the system may be transferred from $x = 0$ at t to $x = x_1$ at t_1 by application of some bounded energy input $u_t(\cdot)$ where the energy bound is independent of t.

Energy is to be defined as

$$E \int_{t_0}^{t_1} u^2(t)\, dt;$$

this definition therefore includes the possibility that $u(t)$ represents a random process and x_1 a random variable.

For finite differential systems, the usual definitions of complete and uniform complete controllability imply $C1$ and $C2$, respectively [5]. Assumption $S1$, expressing the state-variable nature of the parameter x, is certainly satisfied by finite linear differential systems and is also satisfied, for example, by the class of infinite dimensional systems considered in [6].

The following result establishes the converse to Theorem 2 in the first case, that is, when t_0 is finite.

Theorem 3

Let a shaping filter for $r(t, \tau)$ exist for all $t \geq t_0 > -\infty$ and satisfy $S1$ and $C1$. Then $r(t, t)$ is bounded for $t \geq t_0$ if the shaping filter is L₂IBO stable.

Proof: Assumptions $S1$ and $C1$ imply the existence of a finite t_1, an x_1, and $u(t)$ such that

$$f(t, t_0, x_0) = f(t, t_1, x_1) = \int_{t_0}^{t_1} h(t, \tau)u(\tau)\, d\tau$$

for all $t \geq t_1$. The latter equality follows because $f(t, t_1, x_1)$ represents the output effect of the initial condition x_1. From the Schwarz inequality

[1] Here and in succeeding theorems, the existence of a shaping filter implies that (3) has a real-valued solution for the impulse response $h(t, \tau)$.

$$f^2(t, t_0, x_0) \leq \int_{t_0}^{t_1} h^2(t, \tau)\, d\tau \int_{t_0}^{t_1} u^2(\tau)\, d\tau$$

$$\leq \int_{t_0}^{t} h^2(t, \tau)\, d\tau \int_{t_0}^{t_1} u^2(\tau)\, d\tau \leq C \int_{t_0}^{t_1} u^2(\tau)\, d\tau.$$

The last step follows from the stability hypothesis by Theorem 1. For random $u(t)$, we have

$$E[f^2(t, t_0, x_0)] \leq CE \int_{t_0}^{t_1} u^2(\tau)\, d\tau < \infty$$

for all $t \geq t_1$, which implies from (4) that $r(t, t)$ is bounded for all $t \geq t_1$. But by assumption, $r(t, t)$ is finite for t finite. Hence, $r(t, t)$ is bounded for all $t \geq t_0$.

Assumption $C1$ is needed in the proof only to establish a bound on the transient response. If $C1$ is violated, the theorem is generally invalid. Consider for example the covariance $r(t, \tau) = \mathrm{R}e^{(t+\tau)}$ with $R > 0$. A shaping filter may be realized as

$$\dot{x}(t) = 0$$

$$y(t) = x(t)e^t,$$

with $x(t_0) = x_0$, a zero-mean random variable for which $E[x^2] = R$. The system is autonomous (thus it violates $C1$), has an identically vanishing impulse response, and is certainly L_2IBO stable. However

$$E[f^2(t, t_0, x_0)] = \mathrm{R}e^{2t},$$

an unbounded function. This example presents one function in the class of covariance functions considered in [3], which lead to finite dimensional autonomous shaping filter realizations.

We now establish a converse to Theorem 2 for the limiting case in which t_0 approaches $-\infty$.

Theorem 4

Let a shaping filter for $r(t, \tau)$ exist for all t. If the filter satisfies $C2$ and is L_2IBO stable, then

1) $\lim_{\lambda \to -\infty} E[f^2(t, \lambda, x)] = 0$ for all t and x.
2) $r(t, t)$ is bounded for all t.

Proof: Assumption $C2$ implies that for any x and ξ, there exists a $u_\xi(\cdot)$ and λ such that $\lambda - \xi \leq T < \infty$, and

$$f(t, \lambda, x) = \int_{\xi}^{\lambda} h(t, \tau) u_\xi(\tau)\, d\tau,$$

for all $t \geq \lambda$. From the Schwarz inequality,

$$f^2(t, \lambda, x) \leq \int_{\xi}^{\lambda} h^2(t, \tau)\, d\tau \int_{\xi}^{\lambda} u_\xi^2(\tau)\, d\tau.$$

The stability hypothesis and Theorem 1 imply that

$$\int_{\xi}^{\lambda} h^2(t, \tau)\, d\tau \leq \int_{-\infty}^{\lambda} h^2(t, \tau)\, d\tau$$

$$\leq \int_{-\infty}^{t} h^2(t, \tau)\, d\tau \leq C < \infty$$

for all t, and $C2$ implies that

$$E \int_{\xi}^{\lambda} u_\xi^2(\tau)\, d\tau \leq M < \infty$$

for all ξ. Hence

$$E[f^2(t, \lambda, x)] \leq M \int_{-\infty}^{\lambda} h^2(t, \tau)\, d\tau \leq MC < \infty$$

for all $\lambda \leq t$, and

$$\lim_{\lambda \to -\infty} E[f^2(t, \lambda, x)] = 0.$$

This result in conjuction with (4) establishes the boundedness of $r(t, t)$ for all t (and thus the boundedness of $r(t, \tau)$ for all t and τ since $r(t, \tau)$ is a covariance.)

The method used to bound the transient response in the proofs of Theorems 3 and 4 is also applicable to systems subject to other stability criteria such as bounded-input bounded-output stability.

We have shown above that input–output stability of a shaping filter known to exist may be determined directly from the output covariance function. Moreover, detailed knowledge of the impulse response (or state-variable equations) of the shaping filter is not required for the stability test. Since the results presented are independent of the dimension of the system, the shaping filter may contain distributed as well as lumped elements.

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